Complexification of Bohr-Sommerfeld conditions.

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Abstract

The complex version of Bohr-Sommerfeld conditions is proposed. The BPU-construction (see [BPU] or [T1]) is generalized to this complexification. The new feature of this generalization is a spectral curve. The geometry of such curves is investigated.

1 Global structures on subspaces of Lagrangian cycles

Let (M, ω) be a smooth symplectic manifold of dimesion $2n$. It can be considered as a phase space of a classical mechanical system. The lagrangian submanifolds play an especially important role in symplectic geometry. Let $\mathcal{L}$ be a smooth manifold of dimension n and

$$i \colon \mathcal{L} \to M \tag{1.1}$$

be an embedding of $\mathcal{L}$ into M. Then we can consider the subcycle $i(\mathcal{L}) \subset M$ as an orbit in the space $Map(\mathcal{L}, M)$ of smooth maps to M with respect to the natural action of the diffeomorphisms group $\operatorname{Diff}(\mathcal{L})$:

$$(\mathcal{L} \subset M) = Map(\mathcal{L}, M) / \operatorname{Diff}(\mathcal{L}). \tag{1.2}$$

Such a submanifold is called Lagrangian if

$$i^*(\omega) = 0. \tag{1.3}$$

It is easy to see that the normal bundle

$$N_M\mathcal{L} = T^*\mathcal{L} \tag{1.4}$$

-the cotangent bundle of $\mathcal{L}$. By the Darboux-Weinstein theorem a Lagrangian submanifold ($\mathcal{L} \subset M$) has not invariants of embeddings: a small tubular neighborhood of $\mathcal{L}$ can be identified with the neighborhood of the zero section of the cotangent bundle. Moreover, any Lagrangian subcycle in this neighborhood can be identified with a closed 1-form on $\mathcal{L}$. Hence such closed forms define a chart around a submanifold ($\mathcal{L} \subset M$) in the space $\mathcal{L}(M)$ of all Lagrangian subcycles of M. Moreover the tangent space at $\mathcal{L}$

$$T\mathcal{L}(M)_{\mathcal{L}} = \{\alpha \in \Omega(\mathcal{L}) \quad | \quad d\alpha = 0\} = Z^1(\mathcal{L}) \tag{1.5}$$

- the space of closed 1-forms on $\mathcal{L}$.

So besides the smooth type of $\mathcal{L}$, Lagrangian submanifolds look like points of M.

On the other hand we can consider the trivial Hermitian line bundle L_0 on $\mathcal{L}$. Then every closed form $\alpha \in Z^1(\mathcal{L})$ multiplied by $i = \sqrt{-1}$ can be consided as a flat connection on L_0 and the gauge class of this connection defines a character of the fundamental group

$$\chi_\alpha \colon \pi_1(\mathcal{L}) \to U(1) \in H^1(\mathcal{L}, \mathbb{R})/H^1(\mathcal{L}, \mathbb{Z}) = J_{\mathcal{L}}. \tag{1.6}$$

(Let us call the last torus the Jacobian of $\mathcal{L}$).

Thus an infinitesimal deformation $\alpha \in Z^1$ of $\mathcal{L}$ defines the point

$$[\alpha] \in J_{\mathcal{L}} \tag{1.7}$$

and the space of infinitesimal deformation of $\mathcal{L}$ preserving some fixed class (for example, trivial) of flat connections is

$$B^1(\mathcal{L}) = \{\alpha \in Z^1(\mathcal{L}) \quad | \quad \alpha = df\} \tag{1.8}$$

where $f \in C^\infty(\mathcal{L})$. Such deformations of Lagrangian submanifolds are called *isodrastic* deformations (see [Wei2] for the linguistic explanation of this term). So the tangent space to isodrastic deformations

$$T_I\mathcal{L}(M)_{\mathcal{L}} = B^1(\mathcal{L}) = C^\infty(\mathcal{L})/\mathbb{R}. \tag{1.9}$$

To get some interesting structures on the space of all Lagrangian cycles $\mathcal{L}(M)$ we have to equip our Lagrangian submanifolds with some extra structure. We will consider connected Lagrangian cycles only.

Recall that a *half-weithed* Lagrangian submanifold is a pair $(\mathcal{L}, hF)$ where hF is a smooth half-form on $\mathcal{L}$. A half-weighted manifold is *weighted* automatically because $hF^2 = V$ is a volume form on $\mathcal{L}$. Hence we have the double cover

$$sq\colon hW\mathcal{L}(M) \to W\mathcal{L}(M) \tag{1.10}$$

of the space of *weighted* Lagrangian manifolds ramified along

$$\mathcal{L}(M) = \{(\mathcal{L}, hF) \quad | \quad hF = 0\}.$$

Thus the tangent space at $(\mathcal{L}_0, hF_0)$

$$ThW\mathcal{L}(M)_{(\mathcal{L}_0, hF_0)} = Z^1(\mathcal{L}) \oplus C^\infty(\mathcal{L}) \cdot hF_0 \tag{1.11}$$

where the last vector space we consider as a space of half-forms. This space carries the form

$$(f_1 \cdot hF_0, f_2 \cdot hF_0) = \int_{\mathcal{L}} f_1 \cdot f_2 \cdot hF^2. \tag{1.12}$$

The space $hW\mathcal{L}(M)$ of all half-weighted Lagrangian submanifolds of M admits the function

$$v\colon hW\mathcal{L}(M) \to \mathbb{R}, \quad v(\mathcal{L}, hF) = \int_{\mathcal{L}} hF^2. \tag{1.13}$$

Consider level hypersurfaces of this function

$$hW\mathcal{L}(M)_t = v^{-1}(t) \tag{1.14}$$

-the space of half-weighted Lagrangian cycles of volume t.

Thus when we deform $(\mathcal{L}_0, hF_0)$ inside of $hW\mathcal{L}(M)_t$ then by Moser theorem [Wei1] (more precisely, by its paraphrase) each half-weighted submanifold $(\mathcal{L}, hF)$ in the family of deformation is diffeomorphic as a half-weighted manifold with $(\mathcal{L}_0, hF_0)$. For example

$$(ThW\mathcal{L}(M)_0)_{(\mathcal{L}_0, hF_0)} = Z^1(\mathcal{L}) \oplus (hF_0)^\perp \tag{1.15}$$

-the orthogonal subspace with respect to the form (1.8).

On the subspace of isodrastic deformations of a half-weighted submanifold

$$(T_I hW\mathcal{L}(M)_0)_{(\mathcal{L}_0, hF_0)} = B^1(\mathcal{L}) \oplus (hF_0)^{\perp} \tag{1.16}$$

we can define a 2-form by

$$\Omega_0((f_1, hF_1), (f_2, hF_2)) = \int_{\mathcal{L}} (f_1 \cdot hF_2 - f_2 \cdot hF_1) \cdot hF_0. \tag{1.17}$$

It's easy to see that this form is weak non degenerated. Moreover:

Proposition 1.1 *This form is closed.*

We can thus think of the space of isodrastic deformations of half-weighted Lagrangian submanifolds of fixed volume as an infinite-dimensional symplectic manifold.

In the same vein we get the symplectic form on the space

$$IhW\mathcal{L}(M)_t \subset IhW\mathcal{L}(M) \tag{1.18}$$

of isodrastic deformations of half-weighted cycles of volume t (see below).

The next extra structure is coming from

Prequantization

Let us equip our symplectic manifold M with a complex line bundle L carring a Hermitian connection a_L such that its curvature form

$$F_{a_L} = 2\pi i \cdot k \cdot \omega, \quad k \in \mathbb{Z}. \tag{1.19}$$

Such line bundle (L, a_L) is called a *prequantization* bundle of the phase space (M, ω) of a mechanical system and the integer k is called a *level*. Such quadruple

$$(M, \omega, L, a_L)$$

is a phase space of a mechanical system which is ready to be quantized.

For every Lagrangian submanifold $\mathcal{L} \in \mathcal{L}(M)$ the restriction of (L, a_L) is the trivial line bundle with a flat connection. A gauge class of this connection defines a point

$$\alpha(a_L) \in J_{\mathcal{L}} \tag{1.20}$$

on the Jacobian of $\mathcal{L}$ (see (1.7)). So we can distinguish Lagrangian submanifolds by these points.

Definition 1.1 A Lagrangian cycle $\mathcal{L}$ is called *Bohr-Sommerfeld* (BS for short) if

$$\alpha(a_L) = 0$$

(see (1.7), (1.8) and (1.9)).

We get the submanifold

$$BS(a_L) \subset \mathcal{L}(M) \tag{1.21}$$

of Bohr-Sommerfeld with respect to a pair (L, a_L).

This subspace is isodrastic. The tangent space at $\mathcal{L} \in BS(a_L)$ is

$$TBS(a_L)_{\mathcal{L}} = B^1(\mathcal{L}) \tag{1.22}$$

(see (1.8) and (1.9)) and the normal space

$$N_{\mathcal{L}(M)} BS(a_L) = H^1(\mathcal{L}, \mathbb{R}). \tag{1.23}$$

Now consider the space $hW\mathcal{L}(M)_0$ of half-weighted Lagrangian cycles of zero volume (1.14) as a bundle over $\mathcal{L}(M)$. Then $hW\mathcal{L}(M)_1$ is an affine bundle over it. Let us restrict these bundles to the subspace $BS(L, a_L)$. We get the very one space

$$hWBS(a_L)_t \subset hW\mathcal{L}(M)_t \tag{1.24}$$

of half-weighted BS-Lagrangian cycles which is an infinite-dimensional symplectic manifold with respect to the symplectic form Ω (1.17).

Remark Recall that such symplectic form in the case $\mathcal{L} = S^1$ is commonly used in connection with the Korteweg-de Vries equation (see [DKN]). Such symplectic form has also arisen in the hamiltonian formulation of the motion of vortex patches for planar incompressible fluids.

So we can repeat these constructions for Lagrangian submanifolds of this infinite-dimension manifold (1.24). But first we need to find a line bundle L_B on $hWBS(a_L)_t$ with connection A_W such that its curvature form is proportional to Ω (1.17):

$$F_A = 2\pi \cdot K \cdot \Omega, \quad K \in \mathbb{Z}. \tag{1.25}$$

From Lagrangian to Legendrian

Consider the principal $U(1)$-bundle

$$\pi \colon P \to M \tag{1.26}$$

of our prequantization bundle L. The Hemitian connection a_L is given by a 1-form α. This form defines a contact structure on P with the volume form

$$\frac{1}{2}\alpha \wedge d\alpha^n. \tag{1.27}$$

If $\mathcal{L} \in BS(a_L)$ then the restriction $(L, a_L)|_{\mathcal{L}}$ admits a covariant constant section

$$s_{\mathcal{L}} \colon \mathcal{L} \to P|_{\mathcal{L}}. \tag{1.28}$$

Such a section is defined up to the natural $U(1)$-action on P as a principal $U(1)$-bundle.

Moreover, the submanifold

$$s_{\mathcal{L}}(\mathcal{L}) \subset P \tag{1.29}$$

is a *Legendrian* submanifold of the contact manifold P. Such submanifolds of P are called *Planckian*

Let

$$\pi \colon \Lambda(P) \to BS(a_L) \subset \mathcal{L}(M) \tag{1.30}$$

be the space of all such Planckian cycles over all BS-Lagrangian cycles where π is the projection (1.26).

Obviously $\Lambda(P)$ is a principal $U(1)$-bundle over $BS(a_L)$.

Proposition 1.2 *The tangent space of $\Lambda(P)$ at a Planckian submanifold $s_{\mathcal{L}}(\mathcal{L})$*

$$T\Lambda(P)_{s_{\mathcal{L}}(\mathcal{L})} = C^{\infty}(\mathcal{L}).$$

Definition 1.2 The complex line bundle L_B of the principal $U(1)$-bundle $\Lambda(P)$ (1.30) over $BS(L, a_L)$ is called the *Berry bundle*.

Consider the forgetful map

$$F\colon hWBS(a_L) \to BS(a_L) \tag{1.31}$$

as a bundle over $BS(a_L)$ and lift it to the space of Planckian cycles $\Lambda(P)$. We get the space $hW\Lambda(P)$ of half-weighted Planckian cycles and the forgetful map

$$F\colon hW\Lambda(P)) \to \Lambda(P). \tag{1.32}$$

Proposition 1.3 *This forgetful map is the cotangent bundle of* $\Lambda(P)$*:*

$$hW\Lambda(P) = T^*\Lambda(P).$$

We get this statement immediately from (1.11) and (1.12).

Corollary 1.1 *The space* $hW\Lambda(P)$ *is a symplectic infinite-dimensional manifold with respect to the 2-form*

$$\Omega_P = d\alpha$$

where α *is the standard action 1-form on a cotangent bundle.*

Moreover the $U(1)$-action on $\Lambda(P)$ defines the action of $U(1)$ on the cotangent bundle $T^*\Lambda(P) = hW\Lambda(P)$. This action preserves the form Ω_P.

Proposition 1.4 *(1) The function* v *from (1.13) is a moment map for this* $U(1)$*-action.*

(2) The symplectic quotient (the result of the symplectic reduction procedure

$$v^{-1}(t)/U(1) = hWBS(a_L)_t$$

is the space of half-weighted Bohr-Sommerfeld cycles of volume t*.*

Corollary 1.2 *The space* $hWBS(a_L)_t$ *is an infinite-dimensional symplectic manifold with the symplectic 2-form* Ω_t *which is given by the symplectic reduction procedure from the form* Ω_P*. The form* Ω_0 *is given by formula (1.17).*

Summarizing we get

(1) a 1-parameter family of phase spaces

$$(hWBS(a_L)_t, \Omega_t) \tag{1.33}$$

(2) equiped with the Berry line bundle L_B

(3) with principal $U(1)$-bundle

$$\pi\colon F^*\Lambda(P) \to hWBS(a_L)_t$$

lifted from (1.30).

Now we need to find on $hWBS(a_L)_t$ Hermitian connections on the Berry bundle L_B with a curvature Ω_t.

It is easy to check the following

Proposition 1.5 *(1) Every section*

$$S\colon BS(a_L) \to hWBS(a_L)_t$$

defines an Hermitian connection A_S on the Berry line bundle L_B over $BS(a_L)$;

(2) the curvature form of such connection

$$F_{A_S} = 2\pi \cdot i \cdot \Omega_t.$$

A choice of such section S is called a *half-weighting rule*. Thus every half-weight rule defines the prequantised system

$$(BS(a_L)_t, S^*\Omega_t, L_B, A_S) \tag{1.34}$$

(which is ready to be quantized, see (1.19)).

The important half-weighting rule is a choice an admissible (with ω) Riemannian metric and a metaplectic structure on M. Such metric defines the admissible almost complex structure. If this complex structure is integrable then M becomes a *Kahler manifold* and we can switch on strong algebro-geometric methods.

2 Complex structure

To quantize a prequantized system (M, ω, L, a_L) we need to fix upon some *polarization* of it. The complex polarization is a choice of a complex structure I on M such that M_I is a Kähler manifold with Kähler form ω. Then the curvature form of the Hermitian connection a is of type $(1,1)$, hence the line bundle L is a holomorphic line bundle on M_I. Complex quantization provides the space of wave functions

$$\mathcal{H}_I = H^0(L) \tag{2.1}$$

-the space of holomorphic sections of the line bundle L. (see the survey [K]).

Remark Here we will consider the compact case only. In this case the wave function space (2.1) is finite-dimensional. But all our constructions are correct even in non compact case, for example when M is the cotangent bundle of a configurations space where the space of wave functions are infinite-dimensional.

Returning to Proposition 1.3 we can see that the space $hW\Lambda(P)$ admits a complex structure I_P as a cotangent bundle: the tangent space at an half-weighted Planckian $\Lambda \subset P$

$$ThW\Lambda(P)_{\Lambda,hF} = C^\infty(\pi(\Lambda)) \oplus C^\infty(\pi(\Lambda)) = C^\infty_{\mathbb{C}}(\pi(\Lambda)). \tag{2.2}$$

It is easy to check the following

Proposition 2.1 *(1) This complex structure I_P of (2.2) is integrable.*

(2) This complex structure is compatible with the symplectic structure Ω_P (see Corollary 1.1).

Thus our space $hW\Lambda(P)$ is a Kahler infinite-dimensional manifold. Moreover this Kahler structure is invariant with respect to $U(1)$-action on $hW\Lambda(P)$ induced by $U(1)$-action on $\Lambda(P)$. Hence we can switch on the symplectic reduction with the moment map (1.13) (see Proposition 1.4).

Corollary 2.1 *(1) The symplectic quotient*

$$v^{-1}(t) = hWBS(a_L)_t$$

admits a quotient Kahler structure.

(2) *Every quadruple (1.33) of the family of phase spaces is a Kahler manifold and admits a complex polarization given by the symplectic quotient of the complex structure I_P (2.2).*

(3) *The Berry line bundle L_B is holomorphic.*

(4) *For every half-weighting rule*

$$S\colon BS(a_L) \to hWBS(a_L)_t$$

(see 1) of Proposition 1.5) the connection A_S (1.34) is given by the holomorphic structure on the Berry line bundle L_B.

(5) *Every prequantized system of the family (1.34) admits the canonical complex (Kahlerien) polarization and can be quantized.*

Remark We would like accentuate that the complex and Kahlerien structures described here don't depend on a choice of Kahler structure M_I on M.

Recall that $\pi(\Lambda)$ is a BS-Lagrangian cycle in M. This cycle, as any Lagrangian cycle, *can't be contained by an algebraic divisor.* Thus the restriction map

$$res\colon H^0(L) \hookrightarrow C^\infty_{\mathbb{C}}(\pi(\Lambda)) \tag{2.3}$$

is an embedding. Therefore we have the trivial complex subbundle of the holomorphic cotangent space of $hW\Lambda(P)$:

$$0 \to \widetilde{H^0(L)} \to T^*\Lambda(P) \tag{2.4}$$

or the trivial quotient

$$j\colon T\Lambda(P) \to \widetilde{H^0(L)}^* \to 0. \tag{2.5}$$

(Here $\widetilde{V}$ is the trivial vector bundle with a fiber V). This epimorphism looks like a differential of a map to the vector space $\mathcal{H}^* = H^0(L)^*$. It is indeed. Let us fix a metaplectic structure on M_I (see for example section 1 of [T1]). Then, following Bortwick-Paul-Uribe [BPU], we can construct the map

$$BPU\colon hW\Lambda(P) \to \mathcal{H}^* \tag{2.6}$$

with the differential (2.5). The projectivization of this map is

$$\mathbb{P}BPU\colon hWBS(a_L) \to \mathbb{P}\mathcal{H}^*. \tag{2.7}$$

The construction is the following:

(1) our principal bundle P (1.26) becomes the boundary of the unit disc bundle $D \subset L^*$

$$\partial D = P, \quad D \subset L \tag{2.8}$$

where D is a strictly pseudoconvex domain;

(2) there is the Szego orthogonal projector

$$\Pi\colon L^2_{\mathbb{C}}(P) \to H^0(L) \tag{2.9}$$

to the first component of the Hardy space of boundary values of holomorphic functions (in D) which are linear fiberwise;

(3) for every Planckian submanifold $\Lambda \in P$ in $C^\infty_{\mathbb{C}}(\Lambda)$ there is the space of Legendrian distributions of order m associated with Λ which is the Szego projection of the space of distributions conormal to Λ of order $m + \frac{dimM}{2}$ (see 2.1 of section 2 of [BPU]);

(4) a half-form on Λ is identified with the symbol of a Legendrian distribution of order m (see 2.2 of section 2 of [BPU]), thus at the symbolic level all Legendrian distributions look like delta-functions or their derivatives;

(5) for a Legendrian submanifold Λ equipped with a half-form hF we fix the Legendian distribution of order $\frac{1}{2}$ with symbol hF which is the Szego projection of the delta function δ_Λ.

Summarizing

(1) for every Planckian submanifold $\Lambda \subset P$ equipped with a half-form hF we have the vector

$$BPU(\Lambda, hF) = \Pi_{hF}(\delta_\Lambda) \in H^0(L) \tag{2.10}$$

where Π_{hF} is the Szego projection Π to the Hardy space of the distribution with the symbol hF ;

(2) for every Planckian Λ the image $\pi(\Lambda)$ is a Bohr-Sommerfeld Lagrangian and every Planckian with this image is a translation of Λ by $U(1)$-action on P, thus a pair $(\mathcal{L}, hF)$ defines a point of the projectivization

$$BPU(\mathcal{L}, hF) = \mathbb{P}(\Pi_{hF}(\delta_\Lambda)) \in \mathbb{P}H^0(L)^*. \tag{2.11}$$

Actually the Hermitian structure on L defines the Hermitian structure on $H^0(L)$. We can thus identify

$$H^0(L) = \overline{H^0(L)}^*.$$

Our observations are the following

Theorem 2.1 *(1) The BPU-map (2.6) is holomorphic with respect to the complex structure from (2.2).*

(2) The epimorphism j (2.5) is the differential

$$j = dBPU. \tag{2.12}$$

(3) The Berry bundle on $BS(a_L)$ is holomorphic and

$$L_B = BPU^*(\mathcal{O}_{\mathbb{P}H^0(L)^*}(1)). \tag{2.13}$$

(4) Let $\operatorname{rank} H^0(L) = r_L$ and S^{2r_L-1} be the unit sphere in $H^0(L)$. Then the principal $U(1)$-bundle

$$F^*(\Lambda(P)) = BPU^{-1}(S^{2r_L-1}) \tag{2.14}$$

and this equality is equivariant with respect to $U(1)$-action.

(5) Let Ω_{FS} be the Kahler form on $\mathbb{P}H^0(L)$ given by the Fubini-Study metric induced by the identification $H^0(L) = \overline{H^0(L)}^*$. Then on $hWBS(a_L)_t$

$$\Omega_t = BPU^*\Omega_{FS}. \tag{2.15}$$

(6) Let A_{FS} be the connection on the Hopf bundle on $\mathbb{P}H^0(L)^*$ induced by the Fubini-Study metric. Then for any half-weighting rule (see 1) of Proposition 1.5)

$$BPU^*A_{FS}|_{S(BS(a_L))} \tag{2.16}$$

is a connection on the Berry bundle on $BS(a_L)$ with the curvature form Ω_t.

3 Supercycles

There is another way to give any Lagrangian cycle $\mathcal{L}$ some additional structure. This structure is a *supercycle* structure or a *brane* structure with the support $\mathcal{L}$.

Definition 3.1 A pair $(\mathcal{L}, a)$ where a is a flat $U(1)$-connection of the trivial line bundle on $\mathcal{L}$ is called a supercycle (or brane) supported by $\mathcal{L}$.

The trivial line bundle contains the trivial connection given by its covariant derivative

$$\nabla_0 = d. \tag{3.1}$$

Then every connection a on the trivial line bundle can be identify with 1-differential form $i \cdot \alpha \in i \cdot \Omega(\mathcal{L})$. The flatness means that this form is closed.

Let the moduli space of supercycles be $S\mathcal{L}(M)$. The the forgetful map

$$F\colon S\mathcal{L}(M) \to \mathcal{L}(M) \tag{3.2}$$

from this moduli space can be identified with the projection of the tangent bundle.

$$S\mathcal{L}(M) = T\mathcal{L}(M). \tag{3.3}$$

Thus this space admits the natural complex structure: the tangent space

$$TS\mathcal{L}(M)_{(\mathcal{L},a)} = Z^1(\mathcal{L}) \oplus i \cdot Z^1(\mathcal{L}). \tag{3.4}$$

This almost complex structure is constant thus it is *integrable*.

So every infinitesiamal deformation $(\alpha_1, i{\cdot}\alpha_2)$ of a supercycle $(\mathcal{L}, a)$ defines the *complex deformation*

$$a + \alpha_1 + i \cdot \alpha_2 \tag{3.5}$$

of the flat connection a. The complex gauge class of this *complex* connection defines a character of the fundamental group

$$\chi_{a+\alpha_1+i\cdot\alpha_2}\colon \pi_1(\mathcal{L}) \to \mathbb{C}^* \in H^1(\mathcal{L}, \mathbb{C})/H^1(\mathcal{L}, \mathbb{Z}) = J^{\mathbb{C}}_{\mathcal{L}}. \tag{3.6}$$

Recall that there exists the map

$$r\colon J^{\mathbb{C}}_{\mathcal{L}} \to J_{\mathcal{L}}, \quad r(\chi) = \frac{\chi}{|\chi|} \tag{3.7}$$

Definition 3.2 A deformation $(\alpha_1, i \cdot \alpha_2)$ is called *complex isodrastic* deformation of a supercycle $(\mathcal{L}, a)$ if the character (3.7) is constant.

The space of isodrastic infinitesimal deformations of a supercycle $(\mathcal{L}, a)$ is

$$T_I S\mathcal{L}(M)_{(\mathcal{L},a)} = B^1_{\mathbb{C}} = C^\infty_{\mathbb{C}}(\mathcal{L})/\mathbb{C} \tag{3.8}$$

(compair this formula with (1.19)).

Now every prequantization (1.19)-(1.20) equips every Lagrangian cycle $\mathcal{L}$ with a supercycle structure

$$(\mathcal{L}, \alpha_L|_{\mathcal{L}}) \in S\mathcal{L}(M)$$

and we get the section

$$S_{a_L} \colon \mathcal{L}(M) \to S\mathcal{L}(M). \tag{3.9}$$

Every supercycle $(\mathcal{L}, a)$ defines the family of complex flat connections

$$a_L|_{\mathcal{L}} + u \cdot a, \quad u \in \mathbb{C}. \tag{3.10}$$

A complex gauge equivalence class of such complex flat connections is

$$[a_L|_{\mathcal{L}} + u \cdot a] \in J^{\mathbb{C}}_{\mathcal{L}} \tag{3.11}$$

Definition 3.3 A Lagrangian supercycle $(\mathcal{L}, a)$ is called complex Bohr-Sommerfeld ($BS_{\mathbb{C}}$ for short) if there exists $u \in \mathbb{C}$ such that

$$[a_L|_{\mathcal{L}} + u \cdot a] = 0. \tag{3.12}$$

Let

$$BS_{\mathbb{C}}(a_L) \subset S\mathcal{L}(M) \tag{3.13}$$

be the space of all $BS_{\mathbb{C}}$-supercycles. In particular

$$(\mathcal{L}, 0) \quad \text{is} \quad BS_{\mathbb{C}} \quad \Longrightarrow \quad \mathcal{L} \in BS(a_L). \tag{3.14}$$

On the other hand if $\mathcal{L} \in BS(a_L)$ then

$$(\mathcal{L}, a) \quad \text{is} \quad BS_{\mathbb{C}} \quad \Longrightarrow \quad a \in i \cdot B^1(\mathcal{L}). \tag{3.15}$$

So the space $BS_{\mathbb{C}}(a_L)$ contains the subspace

$$BS_{\mathbb{C}}(a_L)_0 = \{(\mathcal{L}, a) \quad | \quad \mathcal{L} \in BS(a_L), \quad a \in i \cdot B^1(\mathcal{L})\}. \tag{3.16}$$

The tangent space of $BS_{\mathbb{C}}(a_L)_0$ at $(\mathcal{L}, a)$ is

$$(TBS_{\mathbb{C}}(a_L)_0)_{(\mathcal{L},a)} = B^1(\mathcal{L})_{\mathbb{C}} = C^{\infty}_{\mathbb{C}}(\mathcal{L})/\mathbb{C}. \tag{3.17}$$

Proposition 3.1 *If $u \neq u'$ and*

$$[a_L|_{\mathcal{L}} + u \cdot a] = [a_L|_{\mathcal{L}} + u' \cdot a] = 0 \tag{3.18}$$

then

$$(\mathcal{L}, a) \in BS_{\mathbb{C}}(a_L)_0$$

On the space $BS_{\mathbb{C}}(a_L)$ there thus exists the function

$$u\colon BS_{\mathbb{C}}(a_L) \to \mathbb{C} \tag{3.19}$$

with level complex hypersurfaces of this function

$$BS_{\mathbb{C}}(a_L)_u \subset BS_{\mathbb{C}}(a_L). \tag{3.20}$$

Remark Recall that the definition of the zero-level is given in (1.16).

Proposition 3.2 *(1) The tangent space of $BS_{\mathbb{C}}(a_L)$ at a supercycle $(\mathcal{L}, a)$ is*

$$TBS_{\mathbb{C}}(L, a_L) = C^{\infty}_{\mathbb{C}}(\mathcal{L}). \tag{3.21}$$

(2) The normal bundle

$$(NBS_{\mathbb{C}}(L, a_L)_u)_{(\mathcal{L},a)} = H^1(\mathcal{L}, \mathbb{C}). \tag{3.22}$$

We can see that this normal bundle doesn't depend on a thus to be $BS_{\mathbb{C}}$ supercycle is a property of a Lagrangian cycle itself.

Corollary 3.1 *The space $BS_{\mathbb{C}}(a_L)$ possess a complex structure and the function u from (3.19) on it is holomorphic.*

$\mathbb{C}^*$-lifting

Consider the principal $\mathbb{C}^*$-bundle

$$\pi\colon P_{\mathbb{C}} \to M \tag{3.23}$$

of the prequantization line bundle L. Our connection a_L is given by the complex 1-form α and the complex 2-form $d\alpha$ defines a complex symplectic structure on $P_{\mathbb{C}}$:

$$(P_{\mathbb{C}}, d\alpha). \tag{3.24}$$

If $(\mathcal{L}, a) \in BS_{\mathbb{C}}(a_L)$ then there exists a covariant constant sections

$$s_{(\mathcal{L},a)}\colon \mathcal{L} \to P_{\mathbb{C}}|_{\mathcal{L}}. \tag{3.25}$$

Such a section is defined up to the natural $\mathbb{C}^*$-action on $P_{\mathbb{C}}$ as on a principal $\mathbb{C}^*$-bundle.

Moreover, the submanifold

$$s_{(\mathcal{L},a)}(\mathcal{L}) \subset P_{\mathbb{C}} \tag{3.26}$$

is an isotropic submanifold with respect to the 2-form from (3.24). Such submanifolds of $P_{\mathbb{C}}$ are called *Planckian* again.

Let

$$\pi\colon \Lambda(P_{\mathbb{C}}) \to BS_{\mathbb{C}}(a_L) \tag{3.27}$$

be the space of all such complex Planckian cycles over all $BS_{\mathbb{C}}$-Lagrangian cycles where π is the projection (3.23).

Obviously (3.27) is a principal $\mathbb{C}^*$-bundle over $BS_{\mathbb{C}}(a_L)$.

Proposition 3.3 *The tangent space of $\Lambda(P_{\mathbb{C}})$ at a Planckian submanifold $s_{\mathcal{L}}(\mathcal{L})$*

$$T\Lambda(P_{\mathbb{C}})_{s_{\mathcal{L}}(\mathcal{L})} = C^{\infty}_{\mathbb{C}}(\mathcal{L}).$$

Definition 3.4 The complex line bundle L_B of the principal $\mathbb{C}^*$-bundle $\Lambda(P_{\mathbb{C}})$ (3.27) over $BS_{\mathbb{C}}(a_L)$ is called the *Berry bundle*.

Consider the forgetful map

$$F\colon BS_{\mathbb{C}}(a_L) \to \mathcal{L}(M) \tag{3.28}$$

as a bundle over $F(BS_{\mathbb{C}}(a_L))$ and lift it to the space of complex Planckian cycles $\Lambda(P_{\mathbb{C}})$. We get the space $S\Lambda(P)$ of complex Planckian supercycles and the forgetful map

$$F\colon S\Lambda(P_{\mathbb{C}})) \to \Lambda(P_{\mathbb{C}}). \tag{3.29}$$

Proposition 3.4 *This forgetful map is the tangent bundle of $\Lambda(P_{\mathbb{C}})$:*

$$S\Lambda(P_{\mathbb{C}}) = T\Lambda(P_{\mathbb{C}}).$$

It is just an interpretation of previous constructions.

Moreover the $\mathbb{C}^*$-action on $\Lambda(P_{\mathbb{C}})$ defines a holomorphic action of $\mathbb{C}^*$ on the tangent bundle $T\Lambda(P_{\mathbb{C}})$.

Proposition 3.5 *(1) The function u (3.19) is a moment map for this $\mathbb{C}^*$-action.*

(2) The complex quotient (the result of the reduction procedure

$$BS_{\mathbb{C}}(a_L)_u$$

is the space of complex Bohr-Sommerfeld cycles of level u.

Corollary 3.2 *The space $BS_{\mathbb{C}}(L, a_L)_u$ is an infinite-dimensional complex manifold.*

Summarizing we get

(1) a 1-complex parameter family of complex spaces

$$(BS_{\mathbb{C}}(a_L)_u) \tag{3.30}$$

(2) equipped with the holomorphic Berry line bundle L_B

(3) with principal $\mathbb{C}^*$-bundle

$$\pi\colon F^*\Lambda(P_{\mathbb{C}}) \to BS_{\mathbb{C}}(a_L)_u.$$

Again let us fix some complex structure I on M such that M_I is a Kähler manifold with Kähler form ω. Then the line bundle L is a holomorphic line bundle on M_I with the space of holomorphic sections $H^0(L)$ (2.1).

Recall that $\pi(\Lambda)$ is a BS-Lagrangian cycle in M. Thus the restriction map

$$res\colon H^0(L) \hookrightarrow C^\infty_{\mathbb{C}}(\pi(\Lambda)) \tag{3.31}$$

is an embedding. Therefore we have the trivial complex subbundle of the holomorphic tangent space of $\Lambda(P_{\mathbb{C}})$:

$$0 \to \widetilde{H^0(L)} \to T\Lambda(P_{\mathbb{C}}). \tag{3.32}$$

This monomorphism defines the space of commuting holomorphic automorphisms: for every $s \in H^0(L)$ and $(\mathcal{L}, a) \in BS_{\mathbb{C}}(a_L)_u$

$$t_s\colon \Lambda(P_{\mathbb{C}}) \to\colon \Lambda(P_{\mathbb{C}}) \tag{3.33}$$

commute with the $\mathbb{C}^*$-action. Thus we have the family of holomorphic automorphisms

$$t_s\colon BS_{\mathbb{C}}(a_L) \to BS_{\mathbb{C}}(a_L) \tag{3.34}$$

preserving the holomorphic Berry line bundle L_B.

Corollary 3.3 *The space $H^0(L)$ acts on the space $H^0(L_B)$ of holomorphic sections of the Berry line bundle. This action is homogeneous with respect to the $\mathbb{C}^*$-action.*

.

Complex structure

Now suppose that our M is equipped with a Riemannian metric g compatible with the symplectic structure ω. For us the important case is when our symplectic manifold (M, ω) is equipped with a compatible complex structure I which is integrable. In this case (M_I, ω) is a Kahler manifold. Then the space $\mathcal{A}^0 = Z^1(\mathcal{L})$ of flat connections on a Lagrangian submanifold $\mathcal{L}$ has the finite dimensional subspace

$$H^1(\mathcal{L}, \mathbb{R})_h = \{i \cdot \alpha | d\alpha = 0, \quad d * \alpha = 0\} \in \mathcal{A}^0_0 \tag{3.35}$$

of harmonic forms with respect g. So the space $S\mathcal{LM}$ of all Lagrangian super cycles contains the subspace

$$S\mathcal{L}(M_I)^h \subset S\mathcal{L}(M) \tag{3.36}$$

of harmonic super cycles (harmonic with respect to the Riemannian metric induced by the compatible complex structure I).

Remark It is quite convenient to code a compatible Riemanian metric on a symplectic manifold by the corresponding almost complex structure. Usually we are considering Kahler manifolds where this almost complex structure I is integrable and the metric is Kahlerian.

The gauge group

$$\mathcal{G} = Map(M, U(1))$$

actes on our space $S\mathcal{L}(M)$ of all Lagrangian supercycles (on the second component of every pair $(\mathcal{L}, a)$). The orbit space

$$QS = S\mathcal{LM}/\mathcal{G} \tag{3.37}$$

is fibered over the space of Lagrangian cycles

$$F_Q \colon QS \to \mathcal{L}(M) \tag{3.38}$$

with finite dimensional fibers. Such fiber over a smooth Lagrangian submanifold $\mathcal{L}$ is

$$F_Q^{-1}(\mathcal{L}) = H^1(\mathcal{L}, \mathbb{R})/H^1(\mathcal{L}, \mathbb{Z}) = J_{\mathcal{L}}. \tag{3.39}$$

is the jacobian of $\mathcal{L}$ (see (1.6). We can say that the fibration F_Q is the *jacobian fibration of the universal Lagrangian cycle.*

The factorization map (3.40) restricted to the subspace (3.39) of harmonic connections gives the surjection

$$U \colon S\mathcal{L}(M_I)^h \to QS. \tag{3.40}$$

Over a smooth $\mathcal{L}$ this map is the universal cover

$$U \colon H^1(\mathcal{L}, \mathbb{R}) \to J_{\mathcal{L}}. \tag{3.41}$$

The intersection

$$BS_{\mathbb{C}}(a_L) \cap S\mathcal{L}(M_I)^h = BS_{\mathbb{C}}(a_L)^h \tag{3.42}$$

is the space of harmonic $BS_{\mathbb{C}}$-cycles. In particular for every value of the function u (3.19) we have the space

$$BS_{\mathbb{C}}(a_L)_u \cap S\mathcal{L}(M_I)^h = BS_{\mathbb{C}}(a_L)^h_u \tag{3.43}$$

of harmonic supercycles of level u. In particular it is easy to see that

Proposition 3.6

$$BS_{\mathbb{C}}(a_L)^h_0 = BS(a_L)$$

Obviously we can map these spaces to QS (see (3.43))

$$U\colon BS_{\mathbb{C}}(a_L)^h_u \to QBS_{\mathbb{C}}(a_L)^h_u \subset QS. \tag{3.44}$$

Over a smooth $\mathcal{L}$ this map is the universal cover (3.44).

Now we suppose that M_I admits a metaplectic structure (see section 1 of [T1]). Then every smooth Lagrangian submanifold admits a metalinear structure. Any Kahler metric g on M defines a half-weighting rule

$$S_g\colon \mathcal{L}(M) \to hW\mathcal{L}(M) \tag{3.45}$$

(see Proposition 1.5) which equips every Lagrangian cycle $\mathcal{L}$ with a half-form hF such that

$$hF^2 = \frac{V_g}{\int_{\mathcal{L}} V_g} \tag{3.46}$$

where V_g is the Riemannian volume form on $\mathcal{L}$.

As any half-weighting rule the section (3.45) defines the identification $T\Lambda(P_{\mathbb{C}} = T^*\Lambda(P_{\mathbb{C}}$ and defines Kahler structures on every complex quotient $BS_{\mathbb{C}}(a_L)_u$ of Proposition 3.5.

Moreover the BPU-construction (2.6)-(2.7) is working. Indeed to switch it on we have to define a Legendrian submanifold of P (2.8) over a Lagrangian submanifold $\mathcal{L} \subset M$. But for $\mathcal{L} \in F(BS_{\mathbb{C}}(a_L)$ we have a lifting

$$s_{\mathcal{L}}\colon \mathcal{L} \to P_{\mathbb{C}}. \tag{3.47}$$

Then the projection

$$\frac{s_{\mathcal{L}}}{|s_{\mathcal{L}}|} : \mathcal{L} \to P \tag{3.48}$$

gives us what we need (see (3.7)).

Now using the BPU-construction we get the map

$$BPU_{\mathbb{C}} : \Lambda(P_{\mathbb{C}}) \to H^0(L)^* \tag{3.49}$$

and the projectivization of it

$$\mathbb{P}BPU_{\mathbb{C}} : BS_{\mathbb{C}}(a_L) \to \mathbb{P}H^0(L)^*. \tag{3.50}$$

Again we can check that

Theorem 3.1 *(1) The map $BPU_{\mathbb{C}}$ (3.49) is holomorphic with respect to the complex structure of Proposition 3.5;*

(2) the epimorphism res^* (see (3.31)) is the differential of $BPU_{\mathbb{C}}$;

(3) the Berry bundle

$$L_B = BPU_{\mathbb{C}}^*(\mathcal{O}_{\mathbb{P}H^0(L)^*}(1)); \tag{3.51}$$

(4) and so on like in Theorem 2.1.

There are two strategies to use this new complex parameter u:

(1) consider the situation when $u = \frac{1}{N}$ is a real rational number

$$\text{and let} \quad N \to \infty;$$

(2) use u as a complex parameter to create get a geometric object- a complex algebraic curve - a “spectral curve”.

In the first case equipping Lagrangian cycles with half-forms and applying the direct BPU-method we get special configurations of states in wave functions spaces $H^0(L^k)$. This gives an integer structure on vector spaces (2.1).

In this paper we are using the second strategy applying these constructions to examples considered in [T1] and [T2].

4 u-curves of real polarizations

Recall that we do not need a complex structure on M to define a real polarization of a quadruple

$$(M, \omega, L, a_L) \tag{4.1}$$

with the condition (1.19) on the curvature of an Hermitian connection a_L. Then a real polarization of this quadruple is a fibration

$$\pi \colon M \to B, \tag{4.2}$$

such that $\omega_{|\pi^{-1}(b)} = 0$ for every point $b \in B$ and for generic b the fibre $\pi^{-1}(b)$ is a smooth Lagrangian. If we consider the pair (M, ω) as a phase space of a mechanical system (see for example [A]) it admits a real polarization if and only if it is completely integrable. In the compact case a generic fiber is a n-torus T^n, $(2n = dim_{\mathbb{R}} M)$ and $dim B = n$. As every family of Lagrangian cycles the family (4.2) defines the Kodaira-Spencer homomorphism

$$KS_b \colon TB_b \to H^1(\pi^{-1}(b), \mathbb{R}) \tag{4.3}$$

at every smooth fiber $\pi^{-1}(b)$. It is a well know fact that if M is compact then every smooth fiber

$$\pi^{-1}(b) = T^{\frac{dim M}{2}}$$

is n-torus where $dim M = 2n$.

We would like to add the following condition to the definition of a real polarization: for every smooth fiber $\pi^{-1}(b) = T^n$ the Kodaira-Spencer map (4.3)

$$KS_b \colon TB_b \to H^1(T^n, \mathbb{R}) \tag{4.4}$$

is an isomorphism.

Under this condition we can restrict the fibration F_Q (3.38) to B. We get the finite-dimensional family

$$F_Q \colon QS \to B \tag{4.5}$$

of Jacobians of fibers of the projection π (4.2).

Now let us switch to a complex structure M_I on M and the corresponding Kahler metric g. For every level $u \in \mathbb{C}$ consider the intersection

$$SB \cap BS_{\mathbb{C}}(a_L)_u \tag{4.6}$$

and the projection of this set to QS over B (4.5). We have to get a finite set of classes of supercycles:

$$F_Q(SB \cap BS_{\mathbb{C}}(a_L)_u) = \{(\mathcal{L}_1, [a_1])_u, ..., (\mathcal{L}_N, [a_N])_u\} \tag{4.7}$$

where $[a_i] \in J_{\mathcal{L}_i}$ (see (3.22)).

When u swept out $\mathbb{C}$ we get a affine algebraic curve

$$\Sigma = \{(\mathcal{L}_i, [a_i])_u\} \subset BS_{\mathbb{C}}(a_{L^k}) \tag{4.8}$$

This curve can be compactified and normalized to a compact algebraic curve

$$\Sigma^k_{\pi,I} \tag{4.9}$$

Definition 4.1 This curve is called a u-curve of the mechanical system (4.1) with respect to a real polarization π (4.2) and a complex polarization I.

There are two possibilities for the behavior of such a curve with respect to the complex parameter u:

(1) either the natural projection

$$u \to \Sigma^k_{\pi,I} \tag{4.10}$$

is the universal cover. Then $\Sigma^k_{\pi,I}$ is an elliptic curve;

(2) or there is the pencil

$$\phi \colon \Sigma^k_{\pi,I} \to \mathbb{P}^1_u \tag{4.11}$$

such that

$$\phi^{-1}(u) = \{(\mathcal{L}_i, [a_i])_u\} \tag{4.12}$$

is the effective divisor (4.7).

In the second case the atributes of this curve are

(1) the pencil

$$|\xi| = \mathbb{P}^1_u \tag{4.13}$$

(may be with base points) which gives the finite cover (4.9).

(2)

$$\phi^{-1}(0) = B \cap BS^k(M, L)$$

is the set of original Bohr-Sommerfeld fibers (see Example 1 of section 1 from [T1]);

(3) the degree of the surjection (2.4)

$$deg\phi = \#(B \cap BS^k(M, L)) \tag{4.14}$$

is the number of Bohr-Sommerfeld fibers of level k of the fibration π (4.2)

(4) the map

$$i_{\pi,I} \colon \Sigma^k_\pi \hookrightarrow F_Q^{-1}(B) \tag{4.15}$$

which for a point $x \in \Sigma^k_{\pi,I}$ over u gives the u-Bohr-Sommerfeld Lagrangian super fiber of our polarization;

(5) the line bundle $L_{\pi,I}$ on $\Sigma^k_{\pi,I}$ the fiber of which over a point $(\mathcal{L}, [a]) \in \Sigma^k_{\pi,I}$ over u is the line

$$\pi^{-1}(\mathcal{L}) \in \Lambda(P_{\mathbb{C}})$$

(see (3.23)).That is this line bundle

$$L_{\pi,I} = resL_B \tag{4.16}$$

is the restriction of the Berry line bundle (see Definition 3.4).

Summarizing let us fix a real polarization π (4.2) of (M, ω) from the quadruple (4.1). Then sending a complex structure I to the curve $\Sigma^k_{\pi,I}$ (4.8) we get the map

$$m\colon \mathcal{M} \to M_g \tag{4.17}$$

of the moduli space of polarized complex structures on M to the moduli space of curves of genus g.

Not many explicit examples can be shown here. Despite this lack we consider applications of these constructions to the subject and examples of paper [T1].

Elliptic curve

The classical theory of theta functions is the first beautiful subject for application of our construction. Let us start with dimension one case.

Let E be an elliptic curve with zero element $o \in E$ and with flat metric g. Then the tangent bundle TA has the standard constant Hermitian structure (that is, the Euclidean metric, symplectic form and complex structure I). The Kähler form ω gives a polarization of degree 1. If we switch to a complex structure we get a phase space of a classical mechanical system

$$(A = T^2, \omega, L = \mathcal{O}_E(o), a) \tag{4.18}$$

just like (4.1). Let us fix a smooth Lagrangian decomposition of E

$$E = T^2 = S^1_+ \times S^1_-. \tag{4.19}$$

Circles of both families are Lagrangian with respect to ω. This decomposition induces the Lagrangian decomposition $H^1(E, \mathbb{Z}) = \mathbb{Z}_+ \times \mathbb{Z}_-$, and the Lagrangian decomposition $E_k = (S^1_+)_k \times (S^1_-)_k$ of the group of points of any order k i. e. the collection of theta structures of any level k (see [Mum]). Thus this Lagrangian decomposition defines the collection of decompositions of the spaces of wave functions

$$\mathcal{H}^k_I = H^0(E, L^k) = \bigoplus_{w \in (\mathbb{Z})^-_k} \mathbb{C} \cdot \theta_w, \quad \text{with} \quad \operatorname{rank} \mathcal{H}^k_I = k, \tag{4.20}$$

where θ_w is the theta function with characteristic w (see [Mum]).

On the other hand the direct product (4.19) gives us a real polarization

$$\pi\colon E \to S^1_- = B. \tag{4.21}$$

Remark that in this case the action coordinates (see for example [A]) are just the flat coordinates on $S^1_- = B$, and under this identification

$$B \cap BS^k = (S^1_-)_k \tag{4.22}$$

is the subgroup of points of order k.

Thus for our u-curve $\Sigma^k_{\pi,I}$ (4.5) in this case

$$\phi^{-1}(0) = (S^1_-)_k. \tag{4.23}$$

Moreover, the embedding $i_{\pi,I}$ (4.8) sends the collection of points

$$\phi^{-1}(0) \subset \Sigma^k_{\pi,I}$$

to $B = S^1_-$ and precisely

$$i_{\pi,I}(\phi^{-1}(0)) = (S^1_-)_k \in S^1_-. \tag{4.24}$$

Now

$$\pi'\colon E' = SQ \to S^1_- = B \tag{4.25}$$

is the mirror partner of E (see the diagram (1.49) from [T2]).

Summarizing we have

Proposition 4.1 *(1) The u-curve $\Sigma^k_{\pi,I}$ of the system (4.18) is an elliptic curve;*

(2) the map (4.15) is the isogeny of order k

$$i_{\pi,I}\colon \Sigma^k_{\pi,I} \to E' \tag{4.26}$$

induced by the isogeny

$$\mu_k\colon S^1_- \to S^1_-.$$

We have thus the first case (4.10);

(3) the map m (4.17)

$$m\colon M_1 \to M_1$$

is the isogeny map along the period S^1_-;

(4) the Berry bundle (4.16) is

$$L_{\pi,I} = i^*_{\pi,I}\mathcal{O}_{E'}(o) \tag{4.27}$$

the line bundle of degree k.

Finally the canonical class of E is zero. Thus E admits a metaplectic structure and every circle admits a metalinear structure. Thus a Kahler metric g on E defines a half-weighting rule (3.45) and $BPU_{\mathbb{C}}$ map

$$BPU_{\mathbb{C}}\colon \Sigma^k_{\pi,I} \to \mathbb{P}\mathcal{H}^k_I \tag{4.28}$$

(see (4.20)). This is the standard embedding by the complete linear system of degree k to the space with the basis (4.20).

5 Non-abelian theory of theta functions

Now we would like to apply these constructions to the theory of non commutative theta-functions (see [T1]). To come to our standard setup consider the (6g-6)-manifold

$$R_g = \mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{SU}(2))/\,\mathrm{PU}(2) \tag{5.1}$$

– the space of classes of SU(2)-representations of the fundamental group of the Riemann surface Σ_g of genus g. This space can be included to the quadruple

$$(R_g, \Omega, L, A_{\mathrm{CS}}) \tag{5.2}$$

where the symplectic form Ω can be constructed directly following W. Goldman or using symplectic reduction arguments as in [RSW] where the determinant line bundle L with the Chern-Simons connection were constructed also. By the construction, the curvature form of this connection is

$$F_{A_{\mathrm{CS}}} = i \cdot \Omega. \tag{5.3}$$

So we are in the very one situation of the Geometric Quantization.

To switch on a complex structure on R_g we have to give a conformal structure I to our Riemann surface Σ of genus g. We get a complex structure on the space of classes of representations R_g such that

$$R_\Sigma = R_g = \mathcal{M}^{\mathrm{ss}} \tag{5.4}$$

is the moduli space of semistable holomorphic vector bundles on Σ. Then the form $F_{A_{\mathrm{CS}}}$ (5.3) is a $(1,1)$-form and the line bundle L admits the unique holomorphic structure compatible with the Hermitian connection A_{CS}. Thus we get the system of wave functions spaces

$$\mathcal{H}_I^k = H^0(R_I, L^k). \tag{5.5}$$

Ranks of these spaces are given by the Verlinde formula (See [B]):

$$rk \quad \mathcal{H}_I^k = \frac{(k+2)^{g-1}}{2^{g-1}} \sum_{n=1}^{k+1} \frac{1}{(\sin(\frac{n\pi}{k+2}))^{2g-2}} \,. \tag{5.6}$$

A real polarization of R_g, Ω is defined by a trinion decomposition of our Riemann surface Σ. Such a decomposition is given by the choice of a maximal collection of disjoint, noncontractible, pairwise nonisotopic smooth circles on Σ. It is easy to see that any such system contains $3g-3$ simple closed circles

$$C_1, \ldots, C_{3g-3} \subset \Sigma_g, \tag{5.7}$$

and the complement is the union of $2g-2$ trinions P_j.

The invariant of such a decomposition is given by its 3-*valent dual graph* $\Gamma(\{C_i\})$, associating a vertex to each trinion P_i, and an edge linking P_i and P_j to a circle C_l such that

$$C_l \subset \partial P_i \cap \partial P_j.$$

The isotopy type of the system (5.7) defines the map

$$\pi_{\{C_i\}} \colon R_g \to \mathbb{R}^{3g-3} \tag{5.8}$$

to the real euclidian space with fixed coordinates $(c_1, \ldots, c_{3g-3})$ such that

$$c_i(\pi_{\{C_i\}}(\rho)) = \frac{1}{\pi} \cos^{-1}(\frac{1}{2} \operatorname{tr} \rho([C_i])) \in [-1,1]. \tag{5.9}$$

where $\{C_i\} = E(\Gamma)$ - the set of edges of our graph. Then (see [JW1]) the map $\pi_{\{C_i\}}$ is a real polarization of the system $(R_g, k\cdot\omega, L^k, k\cdot A_{\mathrm{CS}})$and coordinates c_i are action coordinates for this Hamiltonian system.

Moreover the image of R_g under $\pi_{\{C_i\}}$ is a convex polyhedron

$$\Delta_{\{C_i\}} \subset [0,1]^{3g-3}. \tag{5.10}$$

Beside of this polyhedron our space contains the integer sublattice $\mathbb{Z}^{3g-3} \subset \mathbb{R}^{3g-3}$, and we can consider the " action" torus:

$$T^A = \mathbb{R}^{3g-3}/\mathbb{Z}^{3g-3} \tag{5.11}$$

containing the topological complex

$$\overline{\Delta_{\{C_i\}}} \tag{5.12}$$

which is the image of $\Delta_{\{C_i\}}$ (5.10) in this action torus. Then we have the following description of Bohr-Sommerfeld fibers of level k: first of all the intersection

$$BS^k \cap \overline{\Delta_{\{C_i\}}} = \overline{\Delta_{\{C_i\}}} \cap (T^A)_k \tag{5.13}$$

where $(T^A)_k$ is the subgroup of points of order k of the action-torus.

To describe the inquired subset we consider our 3-valent graph Γ with the set $V(\Gamma)$ of vertexes ($|V(\Gamma)| = 2g-2$) and the set $E(\Gamma)$ of edges ($|E(\Gamma)| = 3g-3$) and the set W_g^k of all functions

$$w\colon E(\Gamma) \to \{0, \frac{1}{k}, ..., \frac{k-1}{k}\} \tag{5.14}$$

on the collection of edges of the 3-valent graph Γ to the collection of $\frac{1}{k}$ integers. Then

$$W_g^k = (T^A)_k.$$

This set contains the subset $W_g^k(\Gamma)$ of functions subjecting to conditions: for any three edges C_l, C_m, C_n meeting at a vertex P_i

(1) $w(C_l) + w(C_m) + w(C_n) \in \frac{2}{k}\cdot\mathbb{Z}$;

(2) $w(C_l) + w(C_m) + w(C_n) \leq 2$;

(3) for any ordering of the triple C_l, C_m, C_n,

$$|w(C_l) - w(C_m)| \leq w(C_n) \leq w(C_l) + w(C_m); \tag{5.15}$$

(4) if an edge C_i separates the graph then

$$w(C_i) \in \frac{2}{k} \cdot \mathbb{Z}$$

Such function w is called an *admissible integer weight of level* k on the graph Γ (see [JW1]). Now the statement of the proposition follows from results of [JW1].

Let us apply our constructions from the previous sections to this situation. We get the u-curve

$$\Sigma^k_{\pi,I} \subset BS_{\mathbb{C}}(a_{L^k}) \tag{5.16}$$

(see Definition 4.1). It is easy to see that here we have the case 2 of the alternative after Definition 4.1. Thus this curve has the full collection of atributes (4.11) - (4.16). Firstly we need two of them:

the embedding (4.15)

$$i_{\pi,I} \colon \Sigma^k_{\pi,I} \to F_Q^{-1}(\Delta_{\{C_i\}}) \tag{5.17}$$

and the complex BPU-map (3.50)

$$\mathbb{P}BPU_{\mathbb{C}} \colon \Sigma^k_{\pi,I} \to \mathbb{P}H^0(L^k)^*. \tag{5.18}$$

Recall that the space $F_Q^{-1}(\Delta_{\{C_i\}})$ (5.16) contains the Geometric Fourier Transformation of L^k

$$GFT(L^k) \subset F_Q^{-1}(\Delta_{\{C_i\}}). \tag{5.19}$$

The intersection

$$GFT(L^k) \cap i_{\pi,I}(\Sigma^k_{\pi,I}) = BS(a_{L^k}) \tag{5.20}$$

-the collection of BS^k-fibers. Moreover

$$BPU_{\mathbb{C}}(GFT(L^k) \cap i_{\pi,I}(\Sigma^k_{\pi,I})) \subset \mathbb{P}H^0(L^k)^* \tag{5.21}$$

is the Bohr-Sommerfeld basis in $H^0(L^k)^*$ (see [T1]).

The combination of map (5.16) and the forgetful map F_Q defines the map

$$F \circ i_{\pi,I} \colon \Sigma^k_{\pi,I} \to \overline{\Delta_{\{C_i\}}} \subset T^A. \tag{5.22}$$

This map induces the Jacobians homomorphism

$$j_{\pi,I} \colon J_{\Sigma^k_\pi} \to T^A. \tag{5.23}$$

Proposition 5.1 *This homorphism of real tori is surjective*

$$J_{\Sigma^k_\pi} \to T^A \to 0$$

It is follows immediately from (5.12)-(5.13).

Proposition 5.2 *There exists the direct decomposition*

$$J_{\Sigma^k_\pi} = T_0 \times T_+^{3g-3} \times T^A$$

on real tori such that the surjection j_π is the composition

$$J_{\Sigma^k_\pi} \to T_+^{3g-3} \times T^A \to T^A$$

Corollary 5.1 *(1) This decomposition gives the decomposition of the subgroup of order k points*

$$(J_{\Sigma^k_{\pi,I}})_k = (T_0)_k \times (T_+^{3g-3})_k \times (T^A)_k$$

(2) and the embedding

$$(T^A)_k \hookrightarrow (0, 0, (T^A)_k) \subset (J_{\Sigma^k_{\pi,I}})_k$$

(3) and hence the embedding

$$W^k_g(\Gamma) \hookrightarrow (J_{\Sigma^k_{\pi,I}})_k.$$

Now our main geometrical observation is as follows

Proposition 5.3 *The collection of non-abelian theta functions with characteristics (elements of Bohr-Sommerfeld basis) of level k corresponds to the subcollection of abelian theta functions with characteristics of the u-curve $\Sigma^k_{\pi,I}$.*

Proposition 5.4 *There exists a standard embedding of the u-curve in the jacobian*

$$\Sigma^k_\pi \hookrightarrow J_{\Sigma^k_{\pi,I}}$$

such that the intersection

$$\Sigma^k_{\pi,I} \cap W^k_g = W^k_g(\Gamma) = \overline{\Delta_{\{C_i\}}} \cap BS(a_{L^k}).$$

We have to stop here the development of a "theory" that isn't accompanied by a large collection of examples yet.

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